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ON THE UNIVERSALITY OF A SEQUENCE OF POWERS MODULO 1

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ABSTRACT. Recently, we proved that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$, there is an increasing sequence of positive integers $(q_n)_{n=1}^{\infty}$ and a number $\alpha > 1$ such that $||\alpha^{q_n} - r_n|| < \delta_n$ for each $n \geqslant 1$. Now, we prove that there are continuum of such numbers α in any interval $I = [a, b]$, where $1 < a < b$, and give some corollaries to this statement.

1. INTRODUCTION

Throughout, we shall denote by $\{x\}$, $\lceil x \rceil$ and $||x||$ the fractional part of a real number x, the least integer which is greater than or equal to x, and the distance from x to the nearest integer, respectively.

In [1], we showed that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$, there exist an increasing sequence of positive integers $(q_n)_{n=1}^{\infty}$ and a number $\alpha > 1$ such that $||\alpha^{q_n} - r_n|| < \delta_n$ for each $n \geqslant 1$.

Now, we will show that there are continuum of such α, so at least one of them is transcendental. We also give some corollaries to this "universality property" of powers. In some sense, if $q_1 < q_2 < q_3 < \dots$ are positive integers, then the subsequence $(\alpha^{q_n})_{n=1}^{\infty}$ of the sequence of powers $(\alpha^n)_{n=1}^{\infty}$ represents the sequence $(r_n)_{n=1}^{\infty}$ modulo 1 with any prescribed "precision". In addition, we relax the condition on q_n. These numbers need not be integers. They can be any positive numbers with "large" gaps between them.

Theorem 1. *Let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive numbers, where $\delta_n \leqslant 1/2$, and let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $I = [a, b]$ is an interval with $1 < a < b$, and suppose M is the least positive integer satisfying $a^{M-1}(a-1) \geqslant \max(10, 2a/(b-a))$. If $(q_n)_{n=1}^{\infty}$ is a sequence of real numbers satisfying $q_1 \geqslant M$ and*

$$q_{n+1} - q_n \geqslant M + 1 + \max(0, \log_a(2.22/(\delta_n(a-1))))$$

for each $n \geqslant 1$, then the interval I contains continuum of numbers α such that the inequality

$$||\alpha^{q_n} - r_n|| < \delta_n$$

holds for each positive integer n.

This theorem will be proved in the next section. In Section 3, we give some corollaries.

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2. Proof of Theorem 1

Without loss of generality we may assume that $r_n \in [0,1)$ for each $n \geqslant 1$. Let $w = (w_n)_{n=1}^{\infty}$ be an arbitrary sequence consisting of two numbers 0 and 1/2. Consider the sequence $(\theta_n)_{n=1}^{\infty}$ defined as $\theta_{2n-1} = r_n$ and $\theta_{2n} = w_n$ for each positive integer n, namely,

$$(\theta_n)_{n=1}^{\infty} = r_1, w_1, r_2, w_2, r_3, w_3, \dots.$$

Let also $\ell_{2n-1} = q_n$ and $\ell_{2n} = q_{n+1} - M$ for each integer $n \geqslant 1$. The inequalities $q_{n+1} - q_n \geqslant M+1$ and $q_1 \geqslant M$ imply that $M \leqslant \ell_1 < \ell_2 < \ell_3 < \dots$ is a sequence of positive numbers satisfying $\ell_{n+1} - \ell_n \geqslant 1$ for each $n \geqslant 1$.

Put $y_0 = a$ and

$$y_n = \left(\lceil y_{n-1}^{\ell_n} \rceil + \theta_n\right)^{1/\ell_n}$$

for $n \geqslant 1$. Since $\theta_n \geqslant 0$ and $\lceil y_{n-1}^{\ell_n} \rceil \geqslant y_{n-1}^{\ell_n}$, we have $y_n \geqslant y_{n-1}$. Thus the sequence $(y_n)_{n=0}^{\infty}$ is non-decreasing. Furthermore, $y_n^{\ell_n} - \theta_n$ is an integer, so $\{y_n^{\ell_n}\} = \{\theta_n\} = \theta_n$ for every $n \in \mathbb{N}$.

From $\lceil y_{n-1}^{\ell_n} \rceil < y_{n-1}^{\ell_n} + 1$ and $\theta_n < 1$, we deduce that $y_n^{\ell_n} = \lceil y_{n-1}^{\ell_n} \rceil + \theta_n < y_{n-1}^{\ell_n} + 2$. Hence $(y_n/y_{n-1})^{\ell_n} < 1 + 2y_{n-1}^{-\ell_n}$. Since $\ell_n > 1$ for every $n \geqslant 1$, we have $y_n/y_{n-1} < 1 + 2y_{n-1}^{-\ell_n}/\ell_n$. This implies that $y_n - y_{n-1} < 2/(\ell_n y_{n-1}^{\ell_n - 1})$. Since $y_n \geqslant y_{n-1} \geqslant \dots \geqslant y_0$ and $\ell_n - \ell_{n-1} \geqslant 1$ for $n \geqslant 2$, by adding n such inequalities (for $y_1 - y_0$, $y_2 - y_1$, $\dots$, $y_n - y_{n-1}$), we obtain

$$y_n - a = y_n - y_0 = \sum_{k=1}^{n} (y_k - y_{k-1}) < \frac{2}{\ell_1} \sum_{k=\ell_1 - 1}^{\infty} y_0^{-k} = \frac{2}{\ell_1 y_0^{\ell_1 - 2}(y_0 - 1)} = \frac{2}{\ell_1 a^{\ell_1 - 2}(a-1)}.$$

Using $a^{M-1}(a-1) \geqslant 2a/(b-a)$ and $\ell_1 = q_1 \geqslant M \geqslant 1$, we deduce that

$$y_n - a < \frac{2}{\ell_1 a^{\ell_1 - 2}(a-1)} \leqslant \frac{2}{a^{\ell_1 - 2}(a-1)} \leqslant \frac{2a}{a^{M-1}(a-1)} \leqslant \frac{2a}{2a/(b-a)} = b - a.$$

Hence $y_n < b$ for every n. Thus the limit $\alpha = \lim_{n\to\infty} y_n$ exists and belongs to the interval $[a,b]$. (Of course, $\alpha = \alpha(w)$ depends on the sequence w.)

Next, we shall estimate the quotient $(y_{k+1}/y_k)^{\ell_n}$ for $k \geqslant n$. Since $(y_{k+1}/y_k)^{\ell_{k+1}} < 1 + 2y_k^{-\ell_{k+1}}$ and $\ell_n/\ell_{k+1} < 1$, we have $(y_{k+1}/y_k)^{\ell_n} < (1 + 2y_k^{-\ell_{k+1}})^{\ell_n/\ell_{k+1}} < 1 + 2y_k^{-\ell_{k+1}}$. It follows that

$$(\alpha/y_n)^{\ell_n} = \prod_{k=n}^{\infty} (y_{k+1}/y_k)^{\ell_n} < \prod_{k=n}^{\infty} (1 + 2y_k^{-\ell_{k+1}})$$

for every fixed positive integer n.

In order to estimate the product $\prod_{k=n}^{\infty}(1+\tau_k)$, where $\tau_k = 2y_k^{-\ell_{k+1}}$, we shall first bound this product from above by $\exp(\sum_{k=n}^{\infty} \tau_k)$ and then use the inequality $\exp(\tau) < 1 + 1.11\tau$, because the sum $\tau = \sum_{k=n}^{\infty} \tau_k$ is less than 1/5. Indeed, using the inequalities $y_k \geqslant y_n \geqslant a$ and $\ell_n - \ell_{n-1} \geqslant 1$, where the inequality is strict for infinitely many n's, we derive that

$$\tau = \sum_{k=n}^{\infty} 2y_k^{-\ell_{k+1}} < \frac{2}{y_n^{\ell_{n+1}-1}(y_n - 1)} \leqslant \frac{2}{a^{\ell_{n+1}-1}(a-1)} \leqslant \frac{2}{a^{\ell_2 - 1}(a-1)}$$

is at most 1/5, because $a^{\ell_2-1}(a-1) \geqslant a^{M-1}(a-1) \geqslant 10$. Consequently,

$$(\alpha/y_n)^{\ell_n} < 1 + 1.11\tau < 1 + 2.22/(y_n^{\ell_{n+1}-1}(y_n-1)).$$

Multiplying both sides by $y_n^{\ell_n}$ and subtracting $y_n^{\ell_n}$ from both sides, we find that

$$0 \leqslant \alpha^{\ell_n} - y_n^{\ell_n} < 2.22/(y_n^{\ell_{n+1}-\ell_n-1}(y_n-1)) \leqslant 2.22/(a^{\ell_{n+1}-\ell_n-1}(a-1)).$$

From this, using $\{y_n^{\ell_n}\} = \theta_n$, we deduce that

$$||\alpha^{\ell_n} - \theta_n|| < 2.22a^{-\ell_{n+1}+\ell_n+1}/(a-1)$$

for each $n \in \mathbb{N}$.

For n odd, the last inequality $||\alpha^{\ell_{2n-1}} - \theta_{2n-1}|| < 2.22a^{-\ell_{2n}+\ell_{2n-1}+1}/(a-1)$ becomes $||\alpha^{q_n} - r_n|| < 2.22a^{-q_{n+1}+q_n+M+1}/(a-1)$. The right hand side is less than or equal to δ_n, because $q_{n+1} - q_n \geqslant M + 1 + \log_a(2.22/(\delta_n(a-1)))$. Thus $||\alpha^{q_n} - r_n|| < \delta_n$ for each $n \in \mathbb{N}$, as claimed.

For n even, the inequality on $||\alpha^{\ell_n} - \theta_n||$ becomes $||\alpha^{\ell_{2n}} - \theta_{2n}|| < 2.22a^{-\ell_{2n+1}+\ell_{2n}+1}/(a-1)$. Using $\ell_{2n+1} = q_{n+1}$, $\ell_{2n} = q_{n+1} - M$, $\theta_{2n} = w_n$ and $a^{M-1}(a-1) \geqslant 10$, we derive that the inequality

$$||\alpha^{q_{n+1}-M} - w_n|| < 2.22a^{-\ell_{2n+1}+\ell_{2n}+1}/(a-1) = 2.22a^{-M+1}/(a-1) \leqslant 0.222$$

holds for each positive integer n.

We shall use this inequality in order to show that all of the numbers $\alpha = \alpha(w) \in I$ corresponding to distinct sequences $w = (w_n)_{n=1}^{\infty}$ of 0 and 1/2 are distinct. Indeed, suppose that $\alpha(w) = \alpha(w')$, although $w_n \neq w_n'$ for some positive integer n. Without loss of generality, we may assume that $w_n = 0$ and $w_n' = 1/2$. Then the inequality $||\alpha(w)^{q_{n+1}-M} - w_n|| < 0.222$ implies that

$$\{\alpha(w)^{q_{n+1}-M}\} \in [0, 0.222) \cup (0.788, 1),$$

whereas the inequality $||\alpha(w')^{q_{n+1}-M} - w_n'|| < 0.222$ implies that

$$\{\alpha(w')^{q_{n+1}-M}\} \in (0.288, 0.722).$$

Consequently, $\alpha(w) \neq \alpha(w')$, as claimed. Since there are continuum of infinite sequences w of two symbols $0, 1/2$, there is continuum of distinct numbers $\alpha(w) \in I$ such that the inequality $||\alpha(w)^n - r_n|| < \delta_n$ holds for each positive integer n. This completes the proof of Theorem 1.

3. Applications of the main theorem

It is well known that there exist many numbers $\alpha > 1$ such that $\lim_{n\to\infty} ||\alpha^n|| = 0$ and, more generally, $\lim_{n\to\infty} ||\xi\alpha^n|| = 0$ for some $\xi \neq 0$. Such α must be a Pisot-Vijayaraghavan number, namely, an algebraic integer whose conjugates over $\mathbb{Q}$ (if any) are all of moduli strictly smaller than 1. (See [3], [4], [5], [6] and also [2].) However, it is knot known whether there is at least one transcendental number $\alpha > 1$ such that $\lim_{n\to\infty} ||\alpha^n|| = 0$ (see [7]). From Theorem 1 we shall derive the following:

Corollary 2. *Let $(q_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty}(q_{n+1}-q_n)=\infty$. Then there is a transcendental number $\alpha>1$ such that $\lim_{n\to\infty}||\alpha^{q_n}||=0$.*

Proof: Let us take $a=11$ and $b=13.2$ in Theorem 1. Then $M=1$. Select $\delta_n=0.222\cdot 11^{2+q_n-q_{n+1}}$. Clearly, $q_{n+1}-q_n=2+\log_{11}(0.222/\delta_n)$, so the condition of the theorem is satisfied. Thus Theorem 1 with $r_1=r_2=r_3=\dots=0$ implies that there exists a transcendental number $\alpha\in[11,13.2]$ such that $||\alpha^{q_n}||<0.222\cdot 11^{2+q_n-q_{n+1}}$ for every positive integer n such that $q_n\geqslant 1$. The condition $\lim_{n\to\infty}(q_{n+1}-q_n)=\infty$ implies that $q_n\geqslant 1$ for all sufficiently large n, and $\lim_{n\to\infty}0.222\cdot 11^{2+q_n-q_{n+1}}=0$. Hence $\lim_{n\to\infty}||\alpha^{q_n}||=0$, as claimed.

Corollary 3. *Let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers, and let $s_1,s_2,s_3,\dots\in\{1,\dots,L\}$, where L is a positive integer. Then, for any $\varepsilon>0$, there is s a transcendental number $\alpha>1$ such that $||s_n\alpha^n-r_n||<\varepsilon$ for each positive integer n.*

Proof: This time, let us take in the theorem $a=2$, $b=3$, $M=5$, $\delta_n=\varepsilon/s_n$ and $q_n=nT$ for each $n\geqslant 1$. Here, T is an integer satisfying $T\geqslant M+1+\log_2(1.11\varepsilon^{-1}L)$. The theorem with each r_n replaced by r_n/s_n implies that there is a transcendental number $\beta\in[2,3]$ such that $||\beta^{Tn}-r_n/s_n||<\varepsilon/s_n$ for each positive integer n. Multiplying by the integer s_n and setting $\alpha=\beta^T$, we get that $||s_n\alpha^n-r_n||<\varepsilon$ for each $n\geqslant 1$, as claimed.

In particular, by Corollary 3, for any real numbers $a\geqslant 0$ and $\varepsilon>0$ satisfying $0\leqslant a<a+\varepsilon\leqslant 1$, there is a transcendental number $\alpha>1$ such that $\{\alpha^n\}\in(a,a+\varepsilon)$ for each positive integer n.

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